A binomial model for valuing equity-linked policies embedding surrender options

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Abstract

The computation of the fair periodical premiums for equity-linked policies in a Cox–Ross–Rubinstein (CRR) [Cox, J.C., et al., 1979. Option pricing: A simplified approach. J. Financial Economics 7, 229–263] evaluation framework is computationally complex. In fact, despite we assume that the equity value evolves according to a CRR lattice, the dynamics of the reference fund made up of equities of the same kind is described by a non-recombining tree since, at each contribution date, a constant contribution is added to the fund value. We propose to overcome this problem by selecting representative values among all the effective reference fund values. Then, the fair periodical premiums for equity-linked policies embedding a surrender option and a minimum guarantee are computed following the usual backward-induction scheme coupled with linear interpolation.

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1. Introduction

Equity-linked policies belong to the class of insurance products that, in the last thirty years, mark the shift from traditional products where the actuarial aspect is predominant to contracts that combine both financial and insurance risks. In these contracts, the financial aspect plays a crucial role and allows insurance companies to be competitive in the modern financial system. Indeed, in an equity-linked policy the policyholder's benefits are linked to the performance of a reference fund made up of equities. Since modern financial theory uses sophisticated techniques for pricing contingent claims, the policy valuation may not be only approached by traditional actuarial techniques.

Generally, an equity-linked policy may be characterized by single or periodical premiums. In the first case, a unique contribution is deemed in the reference fund at the contract stipulation. In the second one, the periodical premiums are typically paid at the beginning of each period at the same dates the deemed contributions in the reference fund are invested.

The dependence of the policy benefits on the reference fund value rises the risk of a possible negative performance of the fund that is entirely borne by the policyholder. This is the reason why, to make the policy more attractive and to provide a lower bound to the insured investment, insurance companies usually insert into the contract a minimum guarantee thus assuming part of that risk.

Among the approximation models proposed to evaluate the fair periodical premiums of equity-linked policies with minimum guarantees, the Brennan and Schwartz (1976) finite-difference approach plays a leading role. Particularly, it is worth evidencing that in a Black–Scholes framework, Brennan and Schwartz (1976) were the first to compute the policy

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premiums employing the financial techniques usually used to price contingent claims. Indeed, the main idea of their model relies on the decomposition suggested for the policy benefit at maturity that may be written as the sum of a sure amount (i.e., the minimum guarantee) plus the value of an immediately exercisable call option on the reference fund with strike price the minimum guarantee, or as the sum of an uncertain amount (i.e., the value of the reference fund) plus the value of an immediately exercisable put option on the reference fund with strike price the minimum guarantee.

Several equity-linked policies also embed a surrender option giving the policyholder the right to terminate early the contract and receive a cash amount (i.e., the surrender value). This option is usually included if the contract provides benefits both in case of death and survival (i.e., endowment policies).

The problem of computing the fair premiums of policies embedding a surrender option may be approached in different ways. We choose to consider the policy contract in a contingent claim framework where the surrender decision is linked to the financial valuation of the contract. In fact, according to this approach, the policyholder exercises the surrender option only when it is financially convenient. Consequently, the whole contract and, in particular, the surrender option are American-style contingent claims whose valuations are obtained by merging together traditional actuarial techniques and financial toolboxes. In this context, it is worth mentioning the contribution of Bacinello (2005) who computed the fair single and periodical premiums of equity-linked endowment policies in a Cox–Ross–Rubinstein (CRR) (Cox et al., 1979) framework.

Computational problems arise when equity-linked policies characterized by periodical premiums are considered. In this case, despite the lattice describing the equity value dynamics is a recombining CRR tree, the presence of periodical deemed contributions makes the lattice describing the evolution of the reference fund not recombining. Indeed, the periodical contribution is a fixed amount added to the reference fund value at each contribution date which kills the recombining effect of the tree at those dates. To overcome this obstacle, in Costabile et al. (2007) we proposed to associate to each node of the tree a certain number of fictitious representative values for the reference fund instead of all the possible ones. The technique used for choosing these representative values is similar to that one developed by Hull and White (1993) for path-dependent options.

In this paper, we propose an algorithm based on selecting at each node of the tree a pre-specified number of effective values of the reference fund. It is worth mentioning that, when each step of the tree describing the dynamics of the reference fund coincides with a contribution date, the number of representative values grows as $n^4/24$, where $n$ is the number of time steps. Then, the usual backward-induction scheme coupled with linear interpolation is used to compute the fair periodical premiums.

The remaining sections are organized as follows. In Section 2, we illustrate the dynamics of the reference fund value when a periodical contribution is deemed in the reference fund showing that our technique allows in overcoming the computational problems arising from not recombining trees. In Section 3, we develop the evaluation model for computing the fair periodical premiums for equity-linked term policies and endowment policies characterized by a surrender option and a minimum guarantee. Then, in Section 4, we illustrate the numerical results while the conclusions are drawn in Section 5.

### 2. The algorithm for the representative reference fund values

In this section, we present the dynamics of the reference fund and the technique of choosing its “representative values”.

The reference fund is generated assuming that a fixed component, $D$, of the periodical premium, $P$, paid at the beginning of each year up to maturity, $T$, is invested yearly to buy equities of the same kind. For the sake of simplicity, we set the contract inception at $t_0 = 0$ and suppose that $T$ is an integer on an annual basis. The equity value evolves in the discrete-time environment described by the CRR model. According to this model, we divide the time to maturity into $n$ time steps each of length $h = T/n$. Without loss of generality, we will choose $n$ as a multiple of $T$ so that the number of time steps between two consecutive premium payment dates, $\Delta = n/T$, is an integer. The equity value at each time step increases by the factor $u = \exp(\sigma \sqrt{h})$ if an up step occurs or decreases by the factor $d = 1/u$ if a down step takes place, where $\sigma$ is the volatility of the equity rate of return. The risk-neutral probability of an up step is $p = (e^{\sigma h} - d)/(u - d)$ while the probability of a down step is $q = 1 - p$ and $r$ is the risk-free interest rate. Finally, $S(i, j) = Su^{i-1}$ is the equity value at node $(i, j)$ ($0 \leq i \leq n$), after $j$ ($0 \leq j \leq i$) up steps and $i - j$ down steps.

In order to show the evolution of the reference fund in this framework, we denote by $t_k = k, k = 0, \ldots, T - 1$, the dates at which the deemed contributions in the reference fund are invested. Let $\mathcal{T} = \{i \in \mathbb{N} \mid 0 \leq i \leq n\}$ be a generic path starting from inception and reaching the node $(i, j)$ characterized by the equity values $S(i, j)$. Following the considered trajectory, at time $t_k = 0$, with the first contribution, $D$, the insurer buys $n(0, 0) = D/\Delta S(0, 0)$ equities. Consequently, for $i = 0$ we set the total number of equities acquired, $N_{\mathcal{T}}(0, 0)$, equal to $n(0, 0)$ and the reference fund has value $N_{\mathcal{T}}(0, 0)S(0, 0) = n(0, 0)S(0, 0)$. For $i > 0$, at each anniversary of the contract, $t_k$, after $k\Delta$ time steps, when the equity value is $S(k\Delta, j_k\Delta)$, the insurer buys $n(k\Delta, j_k\Delta) = D/\Delta S(k\Delta, j_k\Delta)$ equities. Hence, if the equity value follows the

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1 We have to observe that the number of representative values grows most rapidly when $n$ increases. Furthermore, in the case the number of periodical contributions $T$ equals the number of time steps $n$, the tree is not recombining at each step but the computational time is minimized since $n$ cannot be less than $T$.

2 In the context of this article, the term “term policy” refers to a purely financial contract, without mortality risk, in which a periodical premium is paid at the beginning of each period until the contract is in force and the benefit is paid at maturity or when the contract is surrendered.
path $\tau(i, j)$ to reach the node $(i, j)$, the total number of equities acquired by the insurer up to the $i$-th step is given by

$$
N_{\tau(i, j)} = \sum_{k=0}^{\lceil i / \Delta \rceil - 1} n(k\Delta, j_k\Delta),
$$

where $[x]$ is the smallest integer greater than or equal to $x$. It is worth mentioning that $N_{\tau(i, j)}$ is a map that assumes the same value between two consecutive contribution dates. Then, the reference fund value arising from the trajectory $\tau(i, j)$ is

$$
N_{\tau(i, j)}S(i, j) = \sum_{k=0}^{\lceil i / \Delta \rceil - 1} n(k\Delta, j_k\Delta)S(i, j)
= D \sum_{k=0}^{\lceil i / \Delta \rceil - 1} S(i, j) / S(k\Delta, j_k\Delta).
$$

Note that the value $N_{\tau(i, j)}$ is strictly dependent on the path $\tau(i, j)$ considered. It means that, generally, different paths reaching the node $(i, j)$ lead to different values for the total number of equities acquired up to that node. It is also useful to remark that, whenever a surrender option is embedded into the contract, the policyholder has the right to decide whether to escape from the contract or to keep it. We assume that the policyholder may surrender the contract only at each anniversary of the policy, $t_k = k$ (i.e., $i = k\Delta$), just before the payment of the periodical premium. Consequently, at the surrender time, no further contributions are invested into the reference fund and this is the reason why in (1) we take $\lceil i / \Delta \rceil - 1$.

The presence of the constant periodical deemed contribution, $D$, causes a huge increment in the number of possible values of the reference fund since it makes the lattice not recombining and, consequently, the number of nodes grows deeply when the number of time steps increases. The following example further clarifies this point. We fix $S = 100$, $D = 100$, $T = 3$ years, $n = 6$, $\sigma = 0.25$, $r = 0.01$. With this choice, $\Delta = 2$, $h = 0.5$, $u = 1.1934$ and $d = 0.8380$. In Fig. 1, we illustrate the dynamics of the reference fund value. Since $T = 3$ years, there are three contributions invested in the reference fund: the first one at time $t_0 = 0$, the second one at time $t_1 = 1$ year, after two time steps, and the last one at time $t_2 = 2$, after four time steps. With the first contribution, the insurer buys $D/S(0, 0) = 1$ equity and the reference fund value is 100. The dynamics of the reference fund follows a recombining lattice up to the second contribution due at time $t_1 = 1$ year. At that time, the reference fund may assume the values 142.42, 100 and 70.22, respectively. The valuation of the reference fund immediately after the payment of the second contribution, $D$, depends upon the number of equities purchased with that contribution. If the equity has registered two up steps during the first year, i.e., $S(2, 2) = 142.42$, the insurer buys 100/142.42 = 0.7021 equities and the value of the reference fund becomes 242.42. If at time $t_1 = 1$ year only one up step has been registered, the equity price is $S(2, 1) = 100$ and one equity is purchased. Consequently, the value of the reference fund becomes 200. If the first two steps have been down steps, the equity value is $S(2, 0) = 70.22$, the insurer buys 1.4240 equities and the reference fund value becomes 170.22. Clearly, less is the value $S(2, \cdot)$ more is the number of equities acquired by the insurer. Consider now the situation at time $t = 1.5$ years, after two up steps and one down step. At node $(3, 2)$ the equity has price $S(3, 2) = 119.34$ but the value of the reference fund depends upon the path followed by the equity price. Indeed, if the path followed by the equity shows two up steps followed by one down step, the reference fund value is $242.42 \times 0.8380 = 203.14$. On the contrary, if the path of the equity price shows one up step followed by one down step (as well as a down step followed by an up step in this particular case where $\Delta = 2$) and, finally, by another up step, the reference fund value is $200 \times 1.1934 = 238.68$. It is easy to understand that, proceeding forward along the tree in this way, the number of values for the reference fund grows deeply (see the last two steps in Fig. 1) making the problem computationally difficult.
to manage. This is a key point to look at when building up a tree based model to evaluate equity-linked policies since the fast growth of the number of the reference fund values at each time step introduces a huge complexity in the pricing problem.

In order to overcome this obstacle, in Costabile et al. (2007), we used a trick already applied in financial theory to evaluate derivative securities with payoff depending upon a certain function of the asset values that cannot be described by a recombining tree (see Hull and White (1993) and Klassen (2001) for further details). Particularly, we proposed to consider sets of simulated representative reference fund values associated to each node of the lattice instead of considering all the possible values of the reference fund. For each node \((i, j)\) of the tree, this set has been built up by considering a grid of values between the minimum, \(RF_{\min}(i, j)\), and the maximum, \(RF_{\max}(i, j)\), determinations of the reference fund. For \(i = 0\), \(RF_{\max}(0, 0) = RF_{\min}(0, 0) = D\), while for \(i > 0\) when the equity price is \(Su^d\), they are respectively given by

\[
RF_{\min}(i, j) = \sum_{k=0}^{\lceil \frac{i}{\Delta} \rceil - 1} Du_{\min(i-j, i-k\Delta)} u^{\max(i-k\Delta, 0)},
\]

\(0 < i \leq n, 0 \leq j \leq i\)

\[
RF_{\max}(i, j) = \sum_{k=0}^{\lceil \frac{i}{\Delta} \rceil - 1} Du_{\min(i-j, i-k\Delta)} d^{\max(i-k\Delta-j, 0)},
\]

\(0 < i \leq n, 0 \leq j \leq i\).

The elements in the set of the representative values for the node \((i, j)\) have been selected in such a way that the smallest value is \(RF_{\min}(i, j)\), the greatest one is \(RF_{\max}(i, j)\), while the others are of the form \(RF_{\min}(i, j) + a_d\), where \(a\) is a positive real number and \(l\) assumes all the integer values in the interval \([1, l_{\max}(i, j)]\). The integer \(l_{\max}(i, j)\) is determined in order to meet the condition \(RF_{\min}(i, j)e^{\Delta l_{\max}(i, j)} < RF_{\max}(i, j) \leq RF_{\min}(i, j)e^{\Delta l_{\max}(i, j) + 1}\). Once the set of the representative values has been built up for each node, the periodical policy premium is computed by solving a non-linear equation rising from a backward-induction scheme.

In this model, two aspects are needed to be looked into. The first one is related to the choice of the parameter \(a\). This choice represents a crucial point in the implementation since the complexity of the evaluation process is strongly dependent on it. In fact, as \(a\) decreases the number of representative reference fund values associated to each node deeply increases and the computational efficiency of the algorithm diminishes making the premium value very sensitive to the \(a\)-level. In Costabile et al. (2007), we have chosen \(a\) in a way that it allows both the thickening of the number of representative values spanning the interval \([RF_{\min}(i, j), RF_{\max}(i, j)]\) and keeping the algorithm computationally efficient. Definitively, \(a\) has been chosen at the level 0.0001 and all the premium valuations have been done with \(n = 30\). The second aspect is related to linear interpolation that one has to use in the backward recursion when, by moving up (or down) the reference fund value from a generic node \((i, j)\) to the node \((i + 1, j + 1)\) or \((i + 1, j)\), the obtained quantity does not match any representative reference fund value at node \((i + 1, j + 1)\) or \((i + 1, j)\). If this happens, the policy payoff associated to the reference fund onward step is computed by interpolating the policy payoffs associated respectively to the nearest greatest and lower (than that quantity) representative reference fund values at node \((i + 1, j + 1)\) or \((i + 1, j)\).

The choice of a smaller and smaller value for \(a\) assures a fitter and fitter grid of representative reference fund values that reduces the overestimation effect on the premium level due to the required linear interpolation but increases the computational heaviness of the algorithm.

The key difference between the binomial approach proposed here with respect to Costabile et al. (2007) relies upon the technique of choosing the representative reference fund values used to compute the policy premiums. This technique allows us to overcome the limits mentioned above since it considers only “true” representative reference fund values realized on actual paths of the tree. As a consequence, in many cases no interpolation is required for computing the policy value and no dependence on the external parameter to the CRR model, \(a\), exists.

In order to explain the algorithm, let us detail how the set of the representative values for the reference fund is obtained at a generic node \((i, j)\) of the tree \((i = 0, \ldots, n, j = 0, \ldots, i)\), where the equity value is \(S(i, j) = Su^d\). Clearly, at the node \((0, 0)\) we have only one value for the reference fund, namely \(RF(0, 0; 1) = D\). For \(i > 0\), among all the trajectories starting from \((0, 0)\) and reaching \((i, j)\), we choose at first the lowest one, \(\tau_{\min}(i, j)\) (i.e., the path with \(i - j\) down steps followed by \(j\) up steps), and on that we compute the maximum reference fund value, \(RF_{\max}(i, j)\). Since it is the first element in the set of the representative reference fund values for the node \((i, j)\), we set

\[
RF(i, j; 1) = RF_{\max}(i, j) = N_{\tau_{\min}(i, j)}S(i, j)
\]

\[
= \sum_{k=0}^{\lceil \frac{i}{\Delta} \rceil - 1} n(k\Delta, j_k\Delta)S(i, j),
\]

\((k\Delta, j_k\Delta) \in \tau_{\min}(i, j)\).

The second element is defined as follows. Among all the nodes \((k\Delta, j_k\Delta), k = 1, \ldots, \lceil \frac{i}{\Delta} \rceil - 1\) (i.e., in correspondence with the contribution dates \(t_k > 0\), belonging to the first representative trajectory, \(\tau_{\min}(i, j)\), we detect only those ones where the equity has registered the minimum value, \(S_{\min}(k\Delta, j_k\Delta)\). Then, among them, we select the node corresponding to the maximum possible value assumed by \(k\), namely \(k_{\max}\) (i.e., the node \((k_{\max}\Delta, j_{k_{\max}}\Delta)\)), such that the path generated by substituting in \(\tau_{\min}(i, j)\) the node \((k_{\max}\Delta, j_{k_{\max}}\Delta)\) with the node \((k_{\max}\Delta, j_{k_{\max}}\Delta + 1)\) still reaches the node \((i, j)\). If such a \(k\) does not exist, \(\tau_{\min}(i, j)\) is the only representative trajectory and, consequently, there is only one representative reference fund value (computed

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3 The different characterization for \(RF_{\max}(i, j)\) is useful for the algorithm presentation.

4 It is the node \((k\Delta, j_k\Delta)\) associated with \(S_{\min}(k\Delta, j_k\Delta)\) that is the nearest one to the node \((i, j)\).
by (2)) associated to the node \((i, j)\); otherwise, the second representative reference fund value, \(RF(i, j; 2)\), is computed as in (2) by substituting \(\tau_{\text{min}}(i, j; 2)\), is computed as in (2) by substituting \(n(k_{\text{max}} \Delta, j_{\text{max}} \Delta)\), the number of equities acquired on the selected node \((k_{\text{max}} \Delta, j_{\text{max}} \Delta)\) belonging to the path \(\tau_{\text{min}}(i, j)\), with \(n(k_{\text{max}} \Delta, j_{\text{max}} \Delta) d^2\), i.e.,

\[
RF(i, j; 2) = RF(i, j; 1) - \left[ n(k_{\text{max}} \Delta, j_{\text{max}} \Delta) - n(k_{\text{max}} \Delta, j_{\text{max}} \Delta) d^2 \right] S(i, j).
\]

The above relation is due to the following one

\[
n(k_{\text{max}} \Delta, j_{\text{max}} \Delta + 1) = \frac{D}{S(k_{\text{max}} \Delta, j_{\text{max}} \Delta + 1)} = \frac{D}{S(k_{\text{max}} \Delta, j_{\text{max}} \Delta) u^2} = n(k_{\text{max}} \Delta, j_{\text{max}} \Delta) d^2.
\]

Generally, starting from the trajectory \(\tau(i, j)\) that produces the \(l\)-th representative reference fund value, \(RF(i, j; l)\), for the node \((i, j)\), the other representative reference fund values are computed recursively as follows:

- **Step 1:** among all the nodes \((k\Delta, j_k \Delta), k = 1, \ldots, \left[\frac{i}{\Delta}\right] - 1\) (i.e., in correspondence with the contribution dates \(t_k > 0\)), belonging to \(\tau(i, j)\), we detect only those ones where the equity has registered the minimum value, \(S_{\text{min}}(k\Delta, j_k \Delta)\);

- **Step 2:** among them, we select the node corresponding to the maximum possible value assumed by \(k\), namely \(k_{\text{max}}\) (i.e., the node \((k_{\text{max}} \Delta, j_{\text{max}} \Delta)\)), such that the path generated by substituting in \(\tau(i, j)\) the node \((k_{\text{max}} \Delta, j_{\text{max}} \Delta)\) with the node \((k_{\text{max}} \Delta, j_{\text{max}} \Delta + 1)\) still reaches the node \((i, j)\);

- **Step 3:** if such a \(k\) does not exist, \(\tau(i, j)\) is the last representative trajectory and \(RF(i, j; l)\) is the last representative reference fund value associated to the node \((i, j)\) i.e., \(RF_{\text{min}}(c, j)\); otherwise, the \((l + 1)\)-th representative reference fund value is computed by

\[
RF(i, j; l + 1) = RF(i, j; l) - \left[ n(k_{\text{max}} \Delta, j_{\text{max}} \Delta) - n(k_{\text{max}} \Delta, j_{\text{max}} \Delta) d^2 \right] S(i, j).
\]

The following example should further clarify how the algorithm works. Fig. 2 illustrates a binomial tree describing the evolution of the equity value \(S\). We consider a policy that expires after \(T = 3\) years and \(n = 6\). This choice implies \(\Delta = 2\) and, consequently, a constant contribution \(D\) in the reference fund is deemed at time \(i/\Delta, i = 0, 2, 4\). The proposed algorithm considers sets of representative reference fund values defined as follows. Consider the trajectories reaching the node \((6, 3)\). Since there is only one trajectory involved, there is only one reference fund value considered, \(RF(6, 6; 1)\), and it is computed by summing up the number of equities acquired at the \(i\)-th time step, \(i = 0, 2, 4\), \((D/S, D/SD^2, D/SD^4)\) and then by multiplying the result by \(S(6, 6) = Su^6\). For the node \((6, 3)\), the first value of the set, \(RF(6, 3; 1)\), is computed by using the values \((D/S, D/SD^2, D/SD^4)\) on the lowest trajectory \(\tau_{\text{min}}(6, 3)\) and, then, multiplying their sum by \(S(6, 3) = S\).

![Fig. 2. The representative reference fund values.](image)

On the path \(\tau_{\text{min}}(6, 3)\), the minimum possible value for the equity, \(SD^2\), is associated to the nodes \((2, 0)\) and \((4, 1)\) where we purchase \(n(2, 0) = n(4, 1) = D/SD^2\) equities. Following the algorithm presented above, we select the node \((4, 1)\). Then, on the trajectory \(\tau_{\text{min}}(6, 3)\), we replace the node \((4, 1)\) with the node \((4, 2)\) so obtaining a second path that still reaches the node \((6, 3)\). The equity value associated to \((4, 2)\) is given by \(S(4, 2) = S(4, 1)u^2 = S\) and we buy a number of equities equal to \(n(4, 2) = n(4, 1)d^2 = DS\). Consequently, the second representative reference fund value, \(RF(6, 3; 2)\), is computed using the sequence \((D/S, D/SD^2, D/SD^4)\). Following this iterative procedure, the remaining representative values are computed using \((D/S, D/SD^2, D/SD^4)\) and \((D/S, D/SD^2, D/SD^4)\). Finally, the set of the representative reference fund values associated to the node \((6, 3)\) contains all the true reference fund values but those ones generated by \((D/S, D/SD^2)\) (depicted in Fig. 2 through bold lines with terminal arrow) and \((D/S, D/SD^2, D/SD^4)\). In the same way, the set of the representative reference fund values is constructed for the other nodes of the tree.\(^5\)

We close this section with a proposition providing an explicit formula for computing the total number of representative reference fund values, \(\eta(i, j)\), associated to a generic node \((i, j)\) of the lattice that has relevance in the backward-induction scheme presented in the next section. The proof of the proposition is given in the Appendix.

**Proposition.** In a binomial tree characterized by \(n = T \Delta\) time steps, the number of representative reference fund values, \(\eta(i, j)\), associated to a generic node \((i, j), i = 0, \ldots, n,

\(^5\) The sets of the representative reference fund values that we propose is generated starting from the lowest path, \(\tau_{\text{min}}(i, j)\) for the node \((i, j)\). The same sets could be generated in a symmetrical way starting from the highest path, \(\tau_{\text{max}}(i, j)\) (i.e., the path with \(j\) up steps followed by \(i - j\) down steps).
3. Equity-linked premium valuation models

In this section, we present at first the algorithm for computing the fair present value and the fair periodical premium of an equity-linked term policy characterized by a minimum guarantee. Then, we extend the algorithm for computing the periodical premium of an equity-linked term policy with an embedded surrender option and a minimum guarantee and, finally, for an equity-linked endowment policy with the same features. In all the cases, we make the assumption that a fixed amount $D$ of the periodical premium, $P$, paid by the insurer is used to buy equities of the same kind accrued in a reference fund whose value evolves as described in Section 2.

3.1. Equity-linked term policies with minimum guarantee

We start the analysis by considering the case of an equity-linked term policy with minimum guarantee based on periodical premiums, typically paid at the beginning of each year. The presence of the minimum guarantee protects the policyholder’s investment against a negative performance of the reference fund and, consequently, the insurer is forced to pay at least the minimum guaranteed amount. More in details, at maturity, $T$, the policy forces the insurer to pay a capital given by $\max[RF(T), G(T)]$, where $RF(T)$ is the reference fund accrued up to maturity while $G(T)$ is the minimum guarantee at the same date. Among the different types of minimum guarantees, for our numerical valuations provided in Section 4, we consider the case

$$ G(T) = \sum_{z=0}^{T-t} D e^{(T-z)\delta} = D e^{T\delta} - 1, \quad (4) $$

where $\delta > 0$ is the minimum guaranteed continuously compounded interest rate (the case $\delta = 0$ is trivial).

In order to compute the periodical premium, $P$, of such a policy, Brennan and Schwartz (1976) suggested a decomposition for the policy payoff at maturity, $\max[RF(T), G(T)]$, evidencing that it depends upon the payoff of a financial option. In particular, they suggested a put-decomposition as

$$ RF(T) + \max[G(T) - RF(T), 0], \quad (5) $$

and a call-decomposition as

$$ G(T) + \max[RF(T) - G(T), 0]. \quad (6) $$

It is evident that the decomposition in (5) characterizes the payoff at maturity of the equity-linked policy as the sum of the value of the reference fund and a put option written on the reference fund with strike price $G(T)$. Clearly, the put option represents the cost of the minimum guarantee embedded into the contract. Conversely, the decomposition given in (6) characterizes the payoff at maturity as the sum of a fixed amount, $G(T)$, and the payoff of a call option written on the reference fund with strike price $G(T)$.

Following the Brennan–Schwartz analysis, we compute at first the fair present value, $PV$, at time $t_0 = 0$ of the policy
payoff at maturity. If we adopt the decomposition in (5), we obtain
\[
P V = PV_0[R(F(T))] + PV_0[\max(G(T) - R(F(T)), 0)],
\]
where \(PV_0(x)\) indicates the value at time \(t_0 = 0\) of the variable \(x\). In a risk-neutral evaluation framework, this reduces to
\[
P V = De^{rT} \left( \frac{1 - e^{-rT}}{e^r - 1} \right) + De^{-rT} \tilde{E} \{ \max[G(T) - R(F(T)), 0] \},
\]
where \(\tilde{E}\) represents the expected value under the risk-neutral probability measure and \(r\) is the risk-free continuously compounded interest rate. The second term in the sum above is the price at time \(t_0 = 0\) of a put option written on the reference fund with strike price \(G(T)\).

If we consider the decomposition in (6), the value at inception of the policy payoff is given by
\[
P V = PV_0[\max(\Delta)] + PV_0[\max(\Delta G - G(T), 0)]
\]
\[
= De^{T} \left( \frac{e^{T \delta} - 1}{e^r - 1} \right) + De^{-T} \tilde{E} \{ \max[\Delta G - G(T), 0] \}.
\]
The second addendum in the sum above is the price at time \(t_0 = 0\) of a call option written on the reference fund with strike price \(G(T)\).

In order to compute the fair value of the policy at inception, we are left to evaluate at time \(t_0 = 0\) the price of the put option embedded into the contract, as evidenced in (5), or the call option price if we prefer to decompose the policy payoff as in (6). According to the dynamics of the reference fund described in Section 2, we label by \(O(i, j; l), i = 0, \ldots, n, j = 0, \ldots, i, l = 1, \ldots, \eta(i, j)\) the \(l\)-th option value the equity price reaches the node \((i, j)\) and the reference fund value is \(RF(i, j; l)\). At maturity, on the terminal nodes \((n, j)\), in each state of nature \((n, j; l), j = 0, \ldots, n, l = 1, \ldots, \eta(n, j)\), the \(l\)-th put option value is given by
\[
O(n, j; l) = \max(G(T) - RF(n, j; l), 0),
\]
while we consider the call decomposition
\[
O(n, j; l) = \max[RF(n, j; l) - G(T), 0].
\]
Then, for \(i < n\), we compute \(O(i, j; l)\) via the usual backward-induction scheme, i.e.,
\[
O(i, j; l) = e^{-rT}[pO(i + 1, j + 1; l_a) + qO(i + 1, j; l_a)(7)]
\]
(recall that \(p\) is the risk-neutral probability of an up step and \(q = 1 - p\)). The quantities \(O(i + 1, j; l_a)\) and \(O(i + 1, j + 1; l_a)\) are, respectively, the option values corresponding to the reference fund values \([RF(i, j; l) + DI_{t[i=k\Delta,k=1,...,T-1]}]u\) and \([RF(i, j; l) + DI_{t[i=k\Delta,k=1,...,T-1]}]u\). In many cases, these quantities belong to the set of the representative reference fund values associated to the nodes \((i + 1, j)\) and \((i + 1, j + 1)\), respectively. Otherwise, \(O(i + 1, j + 1; l_a)\) can be computed by linear interpolation between the option values \(O(i + 1, j + 1; l_1)\) and \(O(i + 1, j + 1; l_2)\) where \(l_1\) and \(l_2\) are chosen in a way that \(RF(i + 1, j + 1; l_1)\) is the greatest reference fund value smaller than \([RF(i, j; l) + DI_{t[i=k\Delta,k=1,...,T-1]}]u\) and \(RF(i + 1, j + 1; l_2)\) is the smallest value of the reference fund value greater than \([RF(i, j; l) + DI_{t[i=k\Delta,k=1,...,T-1]}]u\). \(O(i + 1, j; l_4)\) is computed analogously.

The fair present value of the policy at inception, \(PV\), is given by the sum of the present value of the two terms arising from the decompositions illustrated in (5) and (6). Then, the fair periodical premium, \(P\), paid typically at the beginning of each year, is simply computed by annualizing at the risk-less interest rate, \(r\), the fair present value.

3.2. Equity-linked term policies with minimum guarantee and surrender option

Now we consider the case of an equity-linked term policy characterized by both minimum guarantee and surrender option. The surrender option embedded in the policy contract gives the policyholder the right to terminate early the contract. We treat the case that the insured may escape from the contract only at the beginning of each year, just before the payment of the periodical premium. If the insured decides to surrender the policy, he receives the surrender value, \(SV\), of the contract that depends upon the current value of the reference fund and upon the minimum guaranteed surrender value, \(G\), evaluated at the surrender time.

Restricting our attention to the representative reference fund values, \(RF(i, j; l)\), we define the corresponding surrender values\(^7\) as
\[
SV(i, j; l) = \max[RF(i, j; l), G(i/\Delta)], \quad i = k\Delta,
\]
\[k = 1, \ldots, T - 1, \quad j = 0, \ldots, i, \quad l = 1, \ldots, \eta(i, j), \quad (8)
\]
\(G(i/\Delta)\) is the minimum guarantee in the case the contract is surrendered at time \(i/\Delta\) and, similar to (4), is defined as
\[
G(i/\Delta) = \sum_{\epsilon=0}^{\epsilon-1} De^{i/(\Delta-\epsilon)\delta}.
\]
(9)

In other words, the insurer is forced to pay at time \(t = i/\Delta\), at least the deemed contributions \(D\) invested at the annual interest rate \(\delta\).\(^8\)

In order to evaluate the periodical policy premium, \(P\), we need to compute the policy value at inception. To this end, let \(V(i, j; l), i = 0, \ldots, n, j = 0, \ldots, i, l = 1, \ldots, \eta(i, j), \) be the value of the policy in the state of nature \((i, j; l)\). At maturity \(T\), after \(n\) time steps, if the contract is still in force, the insurance company is forced to pay
\[
V(n, j; l) = \max[RF(n, j; l), G(T)],
\]
where \(G(T)\) is described by (4).

\(^7\) The surrender value is computed only for the nodes coinciding with the premium payment dates since we hypothesized that the surrender option may be exercised only at those dates.

\(^8\) The minimum guaranteed continuously compounded interest rate \(\delta\) may, in general, assume different values with respect to that one given in (4).
Going backward along the tree, at the $i$-th time step, $0 < i < n$, we must distinguish between the cases of $i$ coinciding with a contribution date and not doing so. If $i$ is a time step coinciding with an anniversary of the contract, i.e., $i = k\Delta, k = 1, \ldots, T - 1$, then in each state of nature $(i, j; l)$ the policyholder may choose the most convenient one between the following two alternatives:

- to exercise the surrender option so as to expire the contract and receive the amount $SV(i, j; l)$;
- to continue the contract by paying immediately the annual premium, $P$.

Hence, the value of the whole contract is given by

$$V(i, j; l) = \max \left[ e^{-rh} [PV(i + 1, j + 1; l_0) + qV(i + 1, j; l_d)] - P, SV(i, j; l) \right].$$

Otherwise, if $i$ is a time step that does not coincide with an anniversary of the policy contract, then

$$V(i, j; l) = e^{-rh} [PV(i + 1, j + 1; l_0) + qV(i + 1, j; l_d)].$$

The quantities $V(i + 1, j; l_d)$ and $V(i + 1, j + 1; l_0)$ are the values of the policy corresponding to the reference fund values $[RF(i, j; l) + Dl_{i=k\Delta, k=1,...,T-1}]$ and $[RF(i, j; l) + Dl_{i=k\Delta, k=1,...,T-1}]$, respectively. In many cases, these quantities belong to the set of the representative reference fund values associated to the nodes $(i + 1, j)$ and $(i + 1, j + 1)$, respectively. Otherwise, $V(i + 1, j + 1; l_0)$ and $V(i + 1, j; l_d)$ are computed by linear interpolation similarly to the previous option values $O(i + 1, j + 1; l_0)$ and $O(i + 1, j; l_d)$.

Following the backward iterative scheme, at time $t_0 = 0$, we are left to compute the fair periodical premium $P$, i.e., to solve the non-linear equation below with respect to $P$

$$V(0, 0; 0) = e^{-rh}[PV(1, 1; l_0) + qV(1, 0; l_d)] - P = 0. \quad (10)$$

We solve Eq. (10) numerically and its solution represents the periodical premium to be paid at the beginning of each year by the insured until he surrenders the contract.\(^9\)

### 3.3. Equity-linked endowment policies with minimum guarantee and surrender option

We devote this section to generalize the previous analysis providing a pricing algorithm for an equity-linked endowment policy with a minimum guarantee and embedding a surrender option. The evaluation framework is easily extended from the previous one by adding two assumptions. The first one relies upon the stochastic independence between the lifetime of the insured and the equity value. The second one is that the valuations made by the insurance company are risk-neutral with respect to mortality.

Consider an equity-linked endowment policy with maturity $T$. At maturity if the insured is still alive, in each state of nature $(n, j; l), j = 0, \ldots, n, l = 1, \ldots, \eta(n, j)$, this policy forces the insurance company to pay $\max\{RF(n, j; l), G(T)\}$, where $G(T)$ has been already defined in (4). Alternatively, if the death happens during the $T$ years of contract, the insurer pays $f_D(i, j; l) = \max\{RF(i, j; l), G(\lfloor i/\Delta \rfloor)\}$, $i = 1, \ldots, n - 1, j = 0, \ldots, l, l = 1, \ldots, \eta(i, j)$, where $f_D(i, j; l)$ is a function specifying the sum paid by the insurer in case of death and

$$G(\lfloor i/\Delta \rfloor) = \sum_{z=0}^\lfloor \frac{i - 1}{\Delta} \rfloor \text{De}^{(i/\Delta-z)^\delta}.$$  

It means that, in case of death during the interval $(i - 1)/\Delta$, $i/\Delta]$, the insured receives at time $i/\Delta$ the maximum between the reference fund and the minimum guarantee evaluated after $i$ time periods. It is worth noting that even more complicated functions specifying the capital to be paid in case of death may be easily managed in this framework.

As usual, to buy the policy the insured pays at the beginning of each year a fixed premium, $P$, and a constant amount $D$ is invested to build up the reference fund. Moreover, the surrender option gives the insured the chance to escape from the contract at the beginning of each year just before the payment of the annual premium.

In order to extend the evaluation framework provided in Section 3.2, for an individual of age $x$, we denote by

- $t \ p_x$, the probability that she/he will survive for at least $t$ years;
- $d \ q_x = 1 - t \ p_x$, the probability of the individual’s death during the next $t$ years.

Once again, our goal is to compute the fair periodical premium, $P$, of the policy. In this case, the capital paid if the insured is alive at maturity $T$, after $n$ time steps, and the contract is still in force is given by

$$V(n, j; l) = \max\{RF(n, j; l), G(T)\},$$

$$j = 0, \ldots, n, \ l = 1, \ldots, \eta(n, j).$$

It means that the insurer is forced to pay the maximum between the minimum guarantee at maturity and the accrued reference fund value at the same time.

Working backward along the tree, at the $i$-th time step ($0 < i < n$), we have to distinguish two cases. Consider, at first, the possibility that $i \neq k\Delta, k = 1, \ldots, T - 1$, i.e., $i$ is a time step that does not coincide with an anniversary of the contract. Then,

$$V(i, j; l) = e^{-rh}[P_t x + i h [PV(i + 1, j + 1; l_0) + qV(i + 1, j; l_d)] + h |d_{x + i h} [Pf_D(i + 1, j + 1; l_0) + qf_D(i + 1, j; l_d)]\],$$

i.e., the policy value at the $i$-th time step is computed following the backward-induction scheme used for pricing financial contingent claims in a risk-neutral world adapted taking into account the benefits paid in the case of death or survival. We remark that $f_D(i + 1, j + 1; l_0)$ (or $f_D(i + 1, j; l_d)$) determines the sum paid by the insurer in case of death when

\(^9\) As already proved by Bacinello (2005), Eq. (10) admits a unique solution.
the reference fund value is $[RF(i, j; l) + DI_{j=k}D_{k=1,...,T-1}]u$ (or $[RF(i, j; l) + DI_{j=k}D_{k=1,...,T-1}]d$) and, only when this value does not belong to the set of the representative values, it is calculated by linear interpolation similarly as before.

Otherwise, if $i = kΔ, k = 1, ..., T - 1$, i.e., $i$ is a time step that coincides with an anniversary of the contract, the policyholder may choose to surrender the contract, so receiving the amount $SV(i, j; l)$ defined in (8), thus $V(i, j; l) = SV(i, j; l)$, or to continue it. In the latter case, the periodical premium, $P$, is paid and the fixed contribution $D$ is deemed in the reference fund to buy equities. Consequently, the policy value is computed as

$$V(i, j; l) = \max \left[ e^{-rh} P_x \left( PV(i + 1, j + 1; l_a) + qV(i + 1, j; l_a) + q_{x+i} \left( pfD(i + 1, j + 1; l_a) + qfD(i, j; l_a) \right) \right] - P, SV(i, j; l) \right].$$

At time $t_0 = 0$, $P$ is the unique solution (see Bacinello (2005) for further details) of the following non-linear equation

$$V(0, 0; 1) = e^{-rh} \left[ P_x \left( PV(1, 1; l_a) + qV(1, 0; l_d) \right) + q_{x+i} \left( pfD(1, 1; l_a) + qfD(1, 0; l_d) \right) \right] - P = 0. \quad (11)$$

We solve (11) numerically and its solution represents the periodical premium to be paid at the beginning of each year by the insured until she/he is alive and has not exercised the surrender option to escape from the contract.

Finally, it is worth mentioning that we model mortality by considering Italian Statistics for Male mortality in 2002. These statistics quoted the annual probabilities of death. It means that, if we consider an individual of age $x$, the table quotes the probability $q_x = q_x$, i.e., the probability that the individual dies before the age $x + 1$. In our evaluation framework, we face the problem to value death probabilities on time periods smaller than one year, $q_{x+i}$. We solve this problem by invoking the assumption of uniformity of the deaths, in the sense that in any fraction of width $z$ of one year it is expected the same fraction $z$ of the deaths related to that age. Hence, the death probability on a fraction $z$ of one year, $q_{x+i}$, equals $q_x$. In our case, $z = h$ and $q_{x+i} = hq_{x+[ih]}$. Clearly $hP_x + i = 1 - hq_{x+i}$.

4. Numerical results

We tested the pricing models presented in Section 3 by computing the fair periodical premiums of different equity-linked policies. Each policy benefit is linked to a reference fund accrued by investing at the beginning of each year a fixed amount $D = 100$.

At first, to assess the goodness of the model, we provide a comparison between the results provided by this model with those ones provided by Brennan and Schwartz (BS) (Brennan and Schwartz, 1976). To this end, we consider at first the case of an equity-linked term policy based on periodical premiums with minimum guarantee but without surrender option. The reasoning of this choice relies upon the fact that Brennan and Schwartz did not take into account the possibility of early withdrawals. In Tables 1 and 2, we report the fair present values ($PV$) and the fair annual premiums ($P$) of fixed-term policies for different values of $n$. The initial equity value is set to $S = 100$, the risk-free continuously compounded interest rate is $r = 0.04$, the volatility is $σ = 0.04$, and the minimum guaranteed interest rate is $δ = 0$. In the last row, we also report the fair present values and the fair annual premium values (CMR) supplied by the model presented in Costabile et al. (2007) (briefly described in Section 2) where we computed all the premiums using $n = 30$ and $a = 0.0001$ since this set of parameters assures a good performance of the algorithm. This is due to the fact that a very small value for $a$ provides for a number of representative reference fund values that well span the range between the

<table>
<thead>
<tr>
<th>$n$</th>
<th>$PV$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>103.5475</td>
<td>477.2878</td>
</tr>
<tr>
<td>100</td>
<td>103.5612</td>
<td>477.2816</td>
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<tr>
<td>150</td>
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<td>477.2838</td>
</tr>
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<td>200</td>
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<tr>
<td>300</td>
<td>103.5704</td>
<td>477.2749</td>
</tr>
<tr>
<td>400</td>
<td>103.5716</td>
<td>477.2737</td>
</tr>
<tr>
<td>500</td>
<td>103.5722</td>
<td>477.2730</td>
</tr>
<tr>
<td>BS</td>
<td>103.5</td>
<td>477.1</td>
</tr>
<tr>
<td>CMR</td>
<td>103.5292</td>
<td>477.2865</td>
</tr>
</tbody>
</table>

Table 2
The values of $PV$ and $P$ for a term policy without surrender option with $T = 15$ years

<table>
<thead>
<tr>
<th>$n$</th>
<th>$PV$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>1176.5372</td>
<td>102.2471</td>
</tr>
<tr>
<td>105</td>
<td>1176.6267</td>
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<tr>
<td>150</td>
<td>1176.6638</td>
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</tr>
<tr>
<td>510</td>
<td>1176.7229</td>
<td>102.2632</td>
</tr>
<tr>
<td>BS</td>
<td>1176.4</td>
<td>102.2</td>
</tr>
<tr>
<td>CMR</td>
<td>1176.2509</td>
<td>102.2222</td>
</tr>
</tbody>
</table>

Table 1
The values of $PV$ and $P$ for term policies without surrender option with $T = 1, 5, 10$ years
maximum and the minimum value of the reference fund at each node so reducing the overestimation effect due to linear interpolation.

To further validate the model, we also provide a comparison with the periodical policy premiums computed by Bacinello (2005) for an endowment policy with a minimum guarantee and embedding a surrender option. In order to allow comparisons to Bacinello (2005) and only for Table 3, we consider the case that the minimum guaranteed interest rate $\delta$ assumes two different values:

- $\hat{\delta}$ to evaluate the minimum guaranteed amount paid in case of death or at maturity;
- $\bar{\delta}$ to evaluate the minimum guaranteed amount paid in case the policyholder decides to exercise the surrender option.

In Table 3, we report the fair annual premium $P$ for different values of $n$, $\hat{\delta}$ and $\bar{\delta}$ for an individual of age $x = 40$ years. The initial equity value is now set to $S = 1$, the risk-free continuously compounded interest rate is $r = 0.05$, the volatility is $\sigma = 0.3$ and the time to maturity of the policy is $T = 20$ years. In the last row, we report the annual premiums computed by Bacinello (BAC) (Bacinello, 2005).

We remark that the difference in the values chosen for $n$ in Tables 1–3 is due to the hypothesis that $n$ must be a multiple of $T$. Furthermore, the difference between the periodical premium values supplied by our model in comparison with those ones provided by Bacinello may be due to two main reasons:

1. we use a different discretization. In fact, while Bacinello considered only the case $n = T$ (i.e., $\Delta = 1$), we take into account also binomial trees based on 400 steps that, in the particular case of a policy with maturity $T = 20$, leads to $\Delta = 20$;
2. we model mortality by considering Italian Statistics for Male mortality in 2002 while Bacinello considered the same mortality statistics referring to 1991. The small changes in mortality during these eleven years may also affect the accuracy of the premium values computed by our model in comparison to Bacinello (2005).

Nevertheless, the periodical premiums supplied by our model are consistent with those ones computed by Bacinello.

In Table 4, we report the fair annual premiums for equity-linked term policies with an embedded surrender option and a minimum guarantee computed for different values of $n$, $r$, $\delta$ and $T$ (recall that $n$ is multiple of $T$). Mortality is not yet considered while the equity value at inception is $S = 100$ and the volatility is $\sigma = 0.1358$.

In Tables 5–7, we report the fair annual premiums for equity-linked endowment policies embedding a surrender option and a minimum guarantee for individuals with different initial ages, $x$. The initial equity value is still $S = 100$.

Comparing the behaviour of the annual premiums, we can note that the risk-less rate influences the premiums, as expected. All the premiums are decreasing when $r$ increases. This is due to the fact that when investing a fixed amount $D$, greater is the equity value less is the number of equities acquired and consequently less is the value of the reference fund. The premiums are also quickly increasing with respect to the minimum interest rate guaranteed $\delta$. It is also possible to note that, when the difference $r - \delta$ remains constant, all the premiums do not change significantly for different values.
of \( r \) and \( \delta \). Furthermore, it is worth noting that the age of the insured does not affect too much the premium values. Contrarily, the maturity of the contract influences the premium values. In particular, all the premiums are increasing. Finally, all the premiums are steeply increasing if we increase the volatility parameter, \( \sigma \).

5. Conclusions

We propose a model for valuing the fair present value and the fair periodical premiums for equity-linked term policies with a minimum guarantee. Then, we extend this model to both equity-linked term policies and endowment policies with a minimum guarantee and embedding a surrender option that gives the policyholder the right to escape from the contract at predetermined dates before the policy maturity. The reference fund evolves in a CRR framework and it is accrued by investing a fixed component of the periodical premiums. The presence of periodical contributions makes the lattice not recombining and, consequently, the pricing problem computationally unmanageable. We propose to overcome this obstacle by choosing sets of representative reference fund values selected among the true reference fund values associated to each node of the tree. The periodical premium is computed by solving a non-linear equation obtained via the usual backward-induction scheme coupled with linear interpolation. We test the proposed model using different types of equity-linked policies and, finally, the obtained numerical results are compared with those ones provided by other existing pricing models.
Acknowledgement

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Appendix. Proof of the proposition

Our first task is to remark that the number of paths reaching a generic node \((i, j)\), detected in order to define the representative reference fund values at that node, is in a one-to-one correspondence with the number of nodes that, at each contribution date, lie between the lowest and the highest node of the tree belonging to the “quadrilateral” with two opposite sides of length \(j\) and the other two of length \(i - j\), respectively, including the highest nodes and excluding the lowest ones. To clarify this aspect, consider a policy with maturity \(T = 10\) years and the situation depicted in Fig. 3 where we illustrate the first \(i = 8\) steps of a binomial tree based on \(n = 20\) steps, thus \(\Delta = 2\). For the node \((i, j) = (7, 4)\), the “quadrilateral” that one has to consider is depicted by thick lines and has vertices \(ABCD\) with sides \(AD\) and \(BC\) of length \(j = 4\) while the sides \(AB\) and \(DC\) have length \(i - j = 3\). The nodes belonging to \(ABCD\) at each contribution date (i.e., those ones that are in a one-to-one correspondence with the number of representative paths obtained with the iterative procedure presented in Section 2) are evidenced by big black circles. In order to count these nodes, we follow this procedure:

**Step 1:** we extend the “quadrilateral” \(ABCD\) up to the contribution epoch immediately after the node \((i, j)\), i.e., \([i/\Delta]\Delta\). This is done in the following way:

1. starting from inception (i.e., from the vertex \(A\) of the tree), we consider the two adjacent sides \(AB\) and \(AD\). Between them, we fix the smallest one, i.e., the side with length \(\min(j, i - j)\), that in Fig. 3 is \(AB\);
2. we extend the other side, \(AD\), by a number of steps equal to \([i/\Delta]\Delta - i\) to obtain the side \(AM\) in Fig. 3.

We consider now the new “quadrilateral” \(ABLM\) with sides of length \(\min(j, i - j)\) and \([i/\Delta]\Delta - \min(j, i - j)\), respectively;

**Step 2:** we will count the nodes belonging to \(ABLM\) corresponding to each contribution date including the nodes lying on \(AM\) and \(ML\) and excluding those ones lying on \(AB\) and \(BL\);

**Step 3:** we will count the nodes belonging to the “quadrilateral” \(CLMD\) corresponding to each contribution date including the nodes lying on \(DM\) and \(ML\) and excluding those ones lying on \(DC\) and \(CL\);

**Final step:** we will obtain the number of nodes belonging to \(ABCD\) by subtracting from the number of nodes belonging to \(ABLM\) the number of nodes belonging to \(CLMD\).

**Step 2**

To count the nodes in the “quadrilateral” \(ABLM\), we divide it into three parts as illustrated in Fig. 3. In terms of the number of nodes on each side, the first and the third part are two congruent “equilateral” triangles, \(AEF\) and \(GLH\), whose side-length detects the number of time steps between the origin and the last contribution date immediately before the \(\min(j, i - j)\)-th step of the tree; the second part, in the middle of the “quadrilateral”, is a six-sided polygon, \(EBGHMF\). Now, in the triangle \(AEF\), we have to take into account, at each contribution date, the nodes enclosed between the opposite sides \(AE\) and \(FA\) including the nodes lying on \(FA\) and excluding those ones lying on \(AE\). Similarly, in the triangle \(GLH\), at each contribution date we count the nodes enclosed between \(GL\) and \(LH\). To this end, we have to know the number of contribution dates falling before the \(\min(j, i - j)\)-th epoch starting from inception. If \(\min(j, i - j) > \Delta\), this number is

\[
\left\lfloor \frac{\min(j, i - j)}{\Delta} \right\rfloor - 1,
\]

and at the first contribution date we will count exactly \(\Delta\) nodes, at the second one we will count \(2\Delta\) nodes and so on. Consequently, the total number of nodes belonging to each triangle is

\[
\sum_{k=1}^{\left\lfloor \frac{i}{\Delta} \right\rfloor - 2} k \Delta = \frac{1}{2} \Delta \left( \left\lfloor \frac{\min(j, i - j)}{\Delta} \right\rfloor - 1 \right) \times \left\lfloor \frac{\min(j, i - j)}{\Delta} \right\rfloor.
\]

It remains to count the number of nodes at each contribution date belonging to the central polygon \(EBGHMF\). Since the sum of the lengths of the two sides \(EB\) and \(BG\) (analogously for \(HM\) and \(MF\)) in terms of steps is

\[
\left\lfloor \frac{i}{\Delta} \right\rfloor \Delta - 2 \Delta \left( \left\lfloor \frac{\min(j, i - j)}{\Delta} \right\rfloor - 1 \right),
\]

then, the number of contribution dates belonging to the central polygon is given by

\[
\left\lfloor \frac{i}{\Delta} \right\rfloor \Delta - 2 \Delta \left( \left\lfloor \frac{\min(j, i - j)}{\Delta} \right\rfloor - 1 \right) + 1.
\]

Now, label by \(EBG\) and \(HM\) the groups of sides (\(EB\)-\(BG\) and \(HM\)-\(MF\), respectively. It is worth noting that \(EBG\) and \(HM\) are opposite sides of the polygon \(EBGHMF\). We observe that the number of nodes at each contribution date between \(EBG\) and \(HM\), including the nodes lying on \(HM\) and excluding

---

10 If there are no more contribution dates after the node \((i, j)\), we extend the “quadrilateral” up to maturity.

11 The nodes corresponding to the vertex \(A\) and \(L\) are considered as belonging to \(AB\) and \(BL\), respectively.

12 The nodes corresponding to the vertex \(D\) and \(L\) are considered as belonging to \(DC\) and \(CL\), respectively.
Fig. 3. The number of representative reference fund values for the node \((i, j)\).

those ones lying on \(E_{BG}\) is exactly \(\min(j, i - j)\). Hence, the number of nodes considered in the central polygon is given by

\[
\min(j, i - j) \left(\left\lceil \frac{i}{\Delta} \right\rceil - 2 \left\lceil \frac{\min(j, i - j)}{\Delta} \right\rceil + 1\right).
\] (12)

The total number of nodes considered for the “quadrilateral” \(ABLM\) is obtained by summing up the number of nodes arising from the two triangles and from the polygon that is

\[
\Delta \left(\left\lceil \frac{\min(j, i - j)}{\Delta} \right\rceil - 1\right) \left\lceil \frac{\min(j, i - j)}{\Delta} \right\rceil + \min(j, i - j) \left(\left\lceil \frac{i}{\Delta} \right\rceil - 2 \left\lceil \frac{\min(j, i - j)}{\Delta} \right\rceil + 1\right).
\]

Finally, if the time step coinciding with the first contribution date is greater than or equal to \(\min(j, i - j)\), the number of nodes considered in the “quadrilateral” \(ABLM\) is simply given by (12).

**Step 3**

In order to compute the number of nodes belonging to the “quadrilateral” \(CLMD\), we start from the node \(D\) in Fig. 3 coinciding with the \(\max(j, i - j)\)-th time step starting from inception. The first contribution date immediately after this node is \(\left\lceil \frac{\max(j, i - j)}{\Delta} \right\rceil + 1\). To count the number of nodes belonging to \(CLMD\) at this date, we have to distinguish between two cases:

- the first contribution date after the node \(D\) falls before the highest vertex of the “quadrilateral” (labeled by \(M\) in Fig. 3).
  In this case, the number of nodes considered at that date coincides with the number of time steps needed to reach the first contribution epoch starting from the node \(D\), that is

  \[
  \Delta \left(\left\lfloor \frac{\min(j, i - j)}{\Delta} \right\rfloor + 1\right) - \max(j, i - j);
  \]

- the contribution date falls after the highest vertex of the “quadrilateral”, \(M\). In this case, the number of nodes considered coincides with

  \[
  \left\lceil \frac{i}{\Delta} \right\rceil \Delta - i,
  \]

i.e., the number of steps needed to reach the contribution date \([i/\Delta]\) starting from \(i\)-th step of the tree (in Fig. 3 it is the length of the sides \(CL\) and \(MD\)).

Consequently, the number of nodes belonging to \(CLMD\) at the first contribution date is given by

\[
\min \left[\Delta \left(\left\lfloor \frac{\max(j, i - j)}{\Delta} \right\rfloor + 1\right) \right.
- \max(j, i - j), \left\lceil \frac{i}{\Delta} \right\rceil \Delta - i\]
\times \min \left(\left\lceil \frac{i}{\Delta} \right\rceil - \left\lfloor \frac{\max(j, i - j)}{\Delta} \right\rfloor - 1, 1\right).
\]
where the second factor in the multiplication above takes into account the possibility that the \((\lceil \max(j, i - j) / \Delta \rceil + 1)\)-th epoch coincides with the \([i / \Delta]\)-th epoch since in this case no node belongs to the “quadrilateral” \(CLMD\).

Now, starting from the \((\lceil \max(j, i - j) / \Delta \rceil + 1)\)-th epoch, the number of contribution dates remaining up to the \((i / \Delta)\)-th epoch is

\[
\left\lfloor \frac{i}{\Delta} \right\rfloor - \left( \left\lfloor \frac{\max(j, i - j)}{\Delta} \right\rfloor + 1 \right) - 1,
\]

but, to take into account the possibility that the \((\lceil \max(j, i - j) / \Delta \rceil + 1)\)-th epoch coincides with the \([i / \Delta]\)-th epoch and no node belongs to the “quadrilateral” \(CLMD\), we have to consider

\[
\max\left( \left\lfloor \frac{i}{\Delta} \right\rfloor - \left\lfloor \frac{\max(j, i - j)}{\Delta} \right\rfloor, 1 \right) - 1.
\]

The number of nodes belonging to the “quadrilateral” \(CLMD\) at each one of these dates is \([i / \Delta] - i\), that is the length of one of the sides of the “quadrilateral” (the side \(CL\) in Fig. 3). This is due to the fact that no contribution arises between the \(i\)-th and the \((i / \Delta)\)-th step of the tree. Consequently, the total number of nodes belonging to the “quadrilateral” \(CLMD\) is

\[
\min\left( \left\lfloor \frac{i}{\Delta} \right\rfloor - \left\lfloor \frac{\max(j, i - j)}{\Delta} \right\rfloor, 1 \right) - 1.
\]

The final step

We recall that the total number of paths generating the representative reference fund values at a generic node \((i, j)\) is obtained by subtracting the number of nodes arising from \(CLMD\) to the number of nodes arising from \(ABLM\) and then adding the lowest path reaching the node \((i, j)\) (i.e., the path with \(i - j\) down steps followed by \(j\) up steps), that is

\[
\eta(i, j) = 1 + \Delta \left( \left\lfloor \frac{\min(j, i - j)}{\Delta} \right\rfloor - 1 \right) + \left( \left\lfloor \frac{\max(j, i - j)}{\Delta} \right\rfloor + 1 \right)
\]

\[
+ \min(j, i - j) \left( \left\lfloor \frac{i}{\Delta} \right\rfloor - 1 \right) \left( \left\lfloor \frac{\min(j, i - j)}{\Delta} \right\rfloor + 1 \right)
\]

\[
- \min\left( \Delta \left( \left\lfloor \frac{\max(j, i - j)}{\Delta} \right\rfloor + 1 \right) \left\lfloor \frac{\min(j, i - j)}{\Delta} \right\rfloor - 1, 1 \right)
\]

\[
\times \min\left( \left\lfloor \frac{i}{\Delta} \right\rfloor - \left\lfloor \frac{\max(j, i - j)}{\Delta} \right\rfloor, 1, 1 \right)
\]

\[
+ \left( \left\lfloor \frac{i}{\Delta} \right\rfloor - 1 \right) \left( \left\lfloor \frac{\min(j, i - j)}{\Delta} \right\rfloor - 1, 1 \right) - 1 \right).
\]

References


